# Proposed beam divergence estimation procedure for the ILRS

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## Abstract

Link budgets for many of the ILRS sites are estimated using divergence values that are derived from the site logs.  Actual data for calculating the station divergence is often incomplete or very optimistic and based on diffraction theory from the full size of the primary mirror for monostatic systems or the full size of the Coude path and beam expander for bistatic systems.

Accurate divergence measurements and a standard method of measuring the divergence is needed by the ILRS for several reasons, including GNSS array requirements and performance prediction and reliable prediction of the energy density delivered on target for the entire ILRS network to deal with requests for information for potential new satellites.

A procedure was developed and presented at the last ILRS meeting for scanning over azimuth and elevation on two satellites and using an equation derived from the laser radar equation for estimating the site divergence.  A new method has been developed which requires scanning a single satellite and performing transmit power measurements to estimate the site divergence.  This method will be presented here with some preliminary results.

# Derivation of analytical divergence estimation expression

- A scan is performed as previously described on a single satellite. The scan is done in AZ and EL up to the points where $N_{pe} \sim 0$ on both sides. The scan endpoints should be noted as well as the center point corresponding to pointing error of 0 degrees.
- At this point, the beam is again centered on the satellite and the full transmit power is recorded. The transmit power is then reduced until $N_{pe} \sim 0$ with 0 pointing error. The transmit power at this point is also recorded.
- The equations for $N_{pe}$ at the scan end (half angle of full scan) and at the reduced transmit power at which pointing error ~ 0 are both equal to an $N_{pe} \sim 0$. By equating these expressions, one of which has pointing error of 0, an expression for the beam divergence can be obtained. This expression is a function of the measured maximum transmit power , the measured transmit power for $N_{pe} \sim 0$, and the measured endpoint of the scan for $N_{pe} \sim 0$. It is also a function of $R_1$ (the range at which the scan is done) and of $R_2$ (the range at which the transmit power is reduced to obtain $N_{pe} \sim 0$). Since the satellite cross section is a function of range, $\sigma(R_1)$ and $\sigma(R_2)$ are also in the expression for beam divergence. If the scan and power reduction measurements can be done quickly so that $R_1 = R_2$, then the expression for beam divergence is a function of the measured power levels and the scan half angle only.
- These relationships allow the derivation of an analytical expression to calculate the divergence from the satellite scan data.

# $N_{pe}$ calculation*

$$N_{pe} = \eta_e * (E_r * \lambda / hc) * \eta_t * G_t * \sigma * (1/4\pi R^2)^2 * A_r * \eta_r * T_a^2 * T_c^2$$

- $\eta_e$ = detector quantum efficiency — Constant
- $E_r$ = laser pulse energy
- $\lambda$ = laser wavelength — Constant
- h = Planck's constant — Constant
- c = speed of light — Constant
- $\eta_t$ = transmit optics efficiency — Constant
- $G_t$ = transmitter gain
- $\sigma$ = satellite optical cross section
- R = slant range to target
- $A_r$ = effective area of receive aperture — Constant
- $\eta_r$ = receive optics efficiency — Constant
- $T_a$ = one-way atmospheric transmission — Constant
- $T_c$ = one-way cirrus cloud transmission — Constant

* "Millimeter Accuracy Satellite Laser Ranging: A Review", John J. Degnan, Contributions of Space Geodesy to Geodynamics: Technology Geodynamics 25, American Geophysical Union, 1993.

# Transmitter Gain*

$$G_t = (8/\theta_t^2) * \exp[-2(\theta/\theta_t)^2]$$

- $\theta_t$ = far field divergence half angle between beam center and $1/e^2$ intensity point
- $\theta$ = beam pointing error; or in this case, the **half angle of the scan**, so that $\theta$ is now $\theta_s$, scan angle.

$$G_{t\text{-scan}} = (8/\theta_t^2) * \exp[-2(\theta_s/\theta_t)^2]$$

For the case where the beam is centered in the scan of the satellite, when transmit power is reduced to the point where $N_{pe} \sim 0$, the pointing error or scan angle is 0 resulting in the simplified expression for $G_{t0}$:

$$G_{t\text{-}0} = (8/\theta_t^2)$$

* "Millimeter Accuracy Satellite Laser Ranging: A Review", John J. Degnan, Contributions of Space Geodesy to Geodynamics: Technology Geodynamics 25, American Geophysical Union, 1993.

# Single Satellite Scan

Assume that scan measurements are taken on a satellite quickly enough that all factors in the expression for $N_{pe}$ are ~ constant with the exception of σ, R, and $G_t$ which changes due to pointing error change. $E_r$ is constant during the scan and is held at $E_{max}$, the typical maximum transmit energy used at the site for ranging. Let the range at which the scan was done be $R_1$ and the cross section at this range $\sigma_1$.

$$N_{pe} = \eta_e * (E_r * \lambda/hc) * \eta_t * G_t * \sigma * (1/4\pi R^2)^2 * A_r * \eta_r * T_a^2 * T_c^2$$

Then the expression for $N_{pe1}$ becomes:

$$N_{pe1} = K * E_{max} * (\sigma_1/R_1^4) * G_{t\text{-}scan}$$

Where K is an approximate constant and

$$G_{t\text{-}scan} = (8/\theta_t^2) * \exp[-2(\theta_s/\theta_t)^2]$$

# Transmit Power Reduction

Assume that transmit power is reduced in a controlled manner that does not change the beam divergence, until the $N_{pe}$ is ~ 0. $E_r$ is reduced to $E_{min}$, and measured and recorded, and the pointing error is held constant at 0. Let the range at which this procedure was done be $R_2$ and the cross section at this range $\sigma_2$.

$$N_{pe} = \eta_e * (E_r * \lambda / hc) * \eta_t * G_t * \sigma * (1/4\pi R^2)^2 * A_r * \eta_r * T_a^2 * T_c^2$$

Then the expression for $N_{pe2}$ becomes:

$$N_{pe2} = K * E_{min} * (\sigma_2 / R_2^4) * G_{t-0}$$

Where K is an approximate constant and

$$G_{t-0} = (8/\theta_t^2)$$

# Equate expressions where $N_{pe} \sim 0$

The equation for $N_{pe1}$, the number of photoelectrons at the end point of the scan is:

$$N_{pe1} = K * E_{max} * (\sigma_1 / R_1^4) * (8/\theta_t^2) * \exp[-2(\theta_s/\theta_t)^2]$$

Where $R_1$ is the range at which the satellite scan was done, and $\sigma_1$ is the satellite cross section at range $R_1$.

The equation for $N_{pe2}$, the number of photoelectrons when transmit power is reduced to $E_{min}$ is:

$$N_{pe2} = K * E_{min} * (\sigma_2 / R_2^4) * (8/\theta_t^2)$$

Where $R_2$ is the range at which the transmit power was done, and $\sigma_2$ is the satellite cross section at range $R_2$.

# Equate expressions where $N_{pe} \sim 0$

$$0 \sim N_{pe1} = N_{pe2} \sim 0$$

$$K * E_{max} * (\sigma_1/R_1^4) * (8/\theta_t^2) * \exp[-2(\theta_s/\theta_t)^2] = K * E_{min} * (\sigma_2/R_2^4) * (8/\theta_t^2)$$

$$\Rightarrow \quad \exp[-2(\theta_s/\theta_t)^2] = [(E_{min}/E_{max}) * (\sigma_2/R_2^4) * (R_1^4/\sigma_1)]$$

$$\Rightarrow \quad \theta_t^2 = -2\theta_s^2/\ln[(E_{min}/E_{max}) * (\sigma_2/R_2^4) * (R_1^4/\sigma_1)]$$

# Divergence Estimate

$$\Rightarrow \quad \theta_t = [-2\theta_s^2/\ln \mathcal{F}(E,R,\sigma)]^{1/2}$$

Half angle of the $1/e^2$ beam divergence.

Where:

$$\mathcal{F}(E,R,\sigma) = [(E_{min}/E_{max})*(\sigma_2/R_2^4)*(R_1^4/\sigma_1)]$$

Since $E_{min}$ and $E_{max}$ are the min and max pulse energies, their ratio is equal to the ratio of the minimum and maximum laser transmit powers.

Ideally, $R_1$ and $R_2$ are ~ equal which implies that σ1 and σ2 are ~ equal so that the expression for $\mathcal{F}$ will reduce to the ratio of the measured transmit powers. If not, $R_1$ and $R_2$ will be recorded while taking measurements and the cross sections can be calculated from knowledge of the range and the LRA construction.

The result is the divergence estimate in terms of known quantities: the satellite range, the measured scan angle, the satellite cross-section and transmit power.

# Divergence Estimate

$$\mathcal{F}(E,R,\sigma) = [(E_{min}/E_{max}) * (\sigma_2/R_2^{\,4}) * (R_1^{\,4}/\sigma_1)]$$

Assume that $R_1$ and $R_2$ are ~ equal which implies that σ1 and σ2 are ~ equal so that the expression for $\mathcal{F}$ will reduce to the ratio of the transmitted pulse energies.

$$\Rightarrow \quad \mathcal{F}(E,R,\sigma) = [E_{min}/E_{max}]$$

Since $E_{min}$ and $E_{max}$ are the min and max pulse energies, their ratio is equal to the ratio of the minimum and maximum laser transmit powers.

$$\Rightarrow \quad \mathcal{F}(E,R,\sigma) = [P_{min}/P_{max}]$$

$$\Rightarrow \quad \theta_t = [-2\theta_s^{\,2}/\ln(P_{min}/P_{max})]^{1/2}$$

# NERC divergence expression

Divergence expression derivation from Jose Rodriguez at NERC using **neutral density filters** at receiver input instead of reducing transmit power. Also assuming that the satellite range and LRA cross-section during measurements were approximately constant.

Once the scan procedure has determined the divergence at the first setting, $\Theta_1$, the beam divergence can be changed and the transmit power (or ND filter value) can be adjusted to again obtain $N_{pe}$ ~ 0 while beam is still centered on the satellite. This allows determination of $\Theta_2$ from $\Theta_1$ by the alternative equation shown to the right in which $T_{ND}$ is the fractional transmission of the filter.

Link budget:

$$N_{pe} = N_{pe}^0 \, \eta \, G_t \, S(r) \, A_r \, T_{ND} \qquad \text{with} \quad G_t(\theta) = \frac{8}{\Theta^2} \exp\left[-2\left(\frac{\theta}{\Theta}\right)^2\right]$$

$N_p^0$: number of photons in a laser pulse
$S(r)$: space losses
$G_t$: transmitter gain for a Gaussian beam
$\theta$: angular distance from beam centre

$\eta$: overall efficiency
$A_r$: receiver area
$T_{ND}$: ND filter transmittance
$\Theta$: half-angle divergence

$$\text{Using:} \quad \left.\begin{array}{l} \theta \quad \text{at which} \quad N_{pe} = 0 \quad (ND = 0) \\ ND \quad \text{at which} \quad N_{pe} = 0 \quad (\theta = 0 \end{array}\right\} \Rightarrow G_t = T_{ND} \, G_t'$$

Reordering:

$$\Theta = \sqrt{\frac{-2\theta^2}{\ln T_{ND}}}.$$

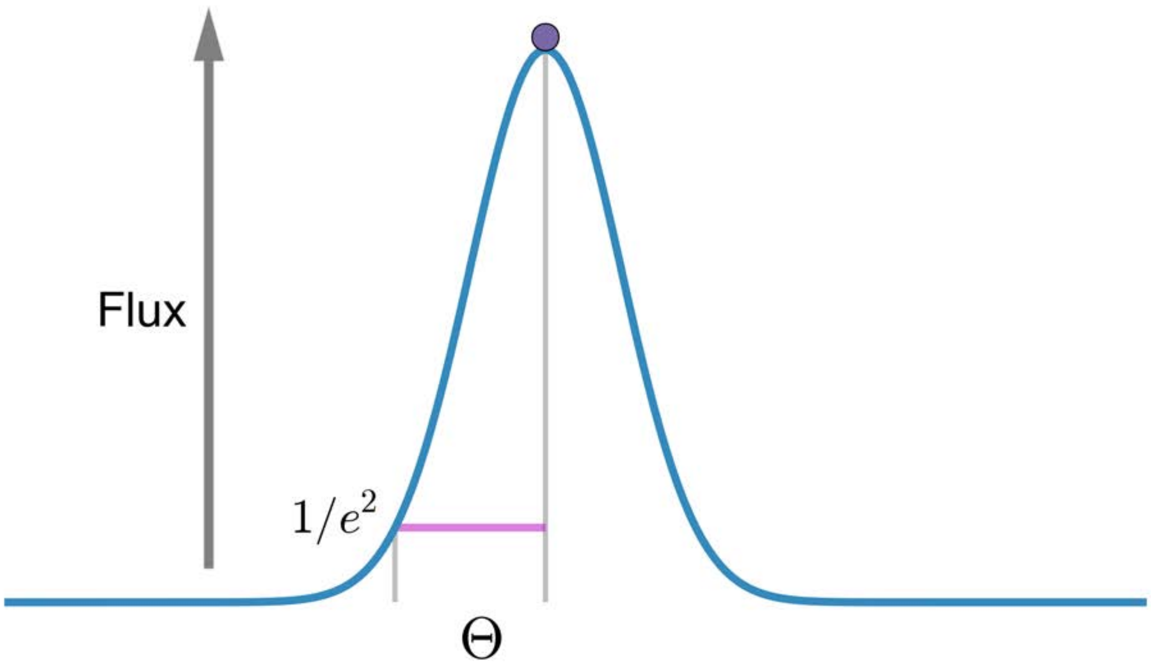
Aternatively:

$$\Theta_2 = \sqrt{\frac{T_{ND,2}}{T_{ND,1}} \Theta_1^2}.$$

# Procedure & Data Required

***Acquire and find signal peak***

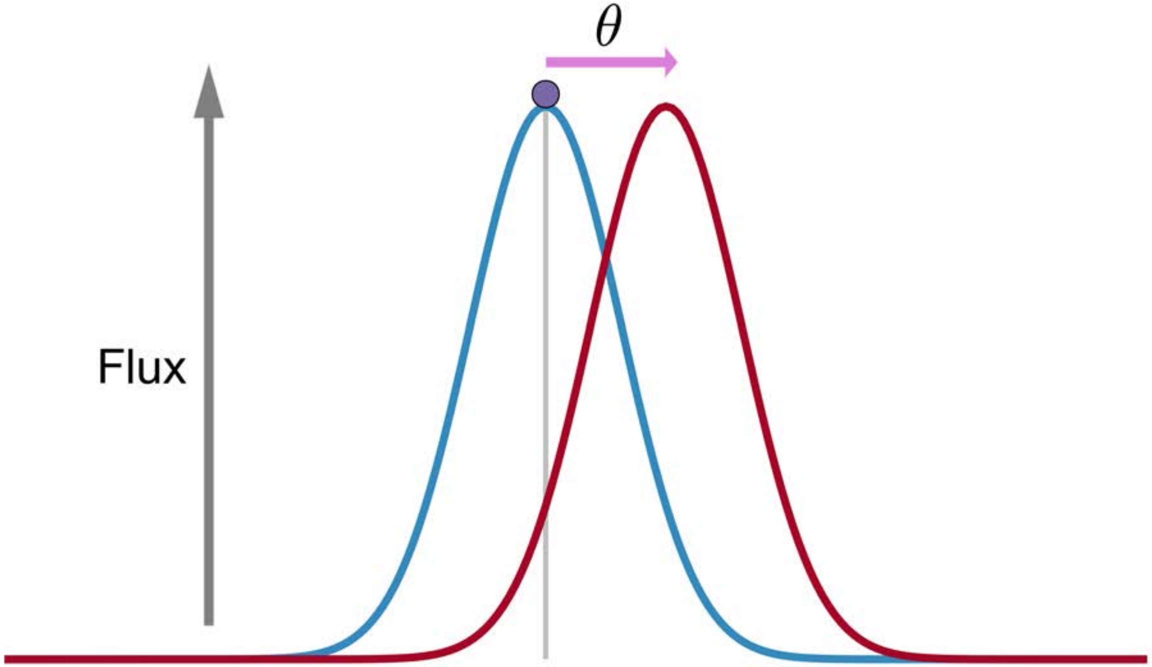
➢ ***Goal is to determine $1/e^2$ divergence half angle, $\Theta$***



# Procedure & Data Required

***Scan in azimuth & elevation  - Initial***

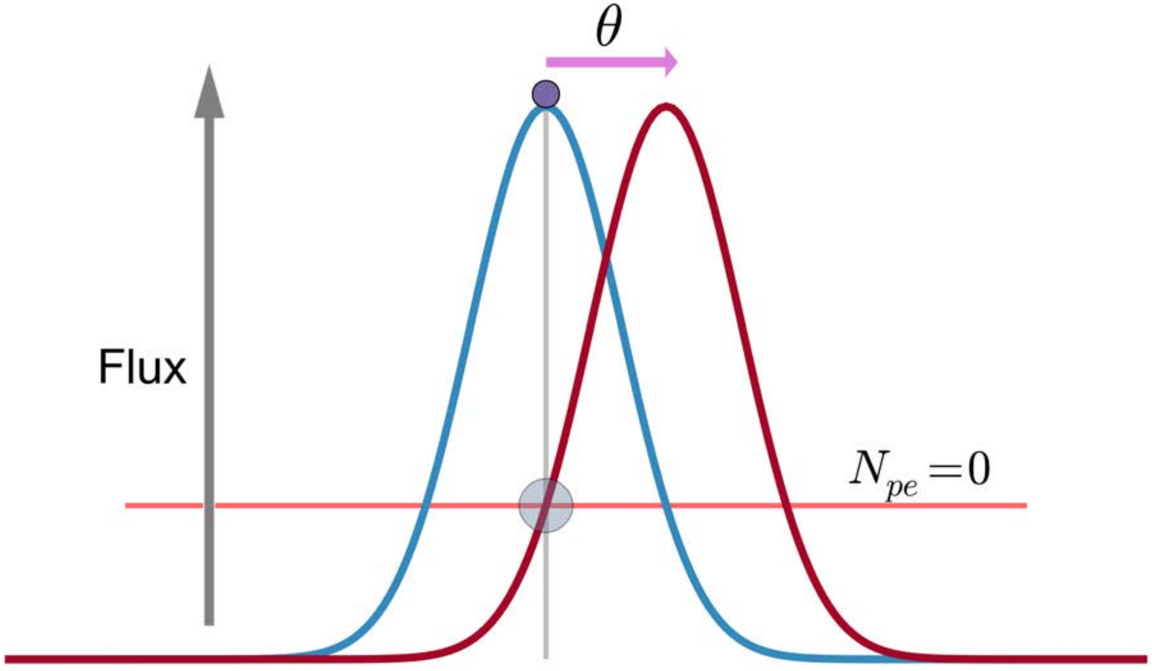
- **Find left and right boundary where the signal is ~ 0**
- **Go past by 2 or 3 steps to confirm**
- **Record offsets and center**
- **Set to center**



# Procedure & Data Required

***Scan in azimuth & elevation to Boundary where Npe ~ 0***

- **Record offsets and center in azimuth before elevation scan**
- **Record range and elevation when scans were done**



# Procedure & Data Required

***After scan, center the beam at the AZ & EL center determined during previous scanning***

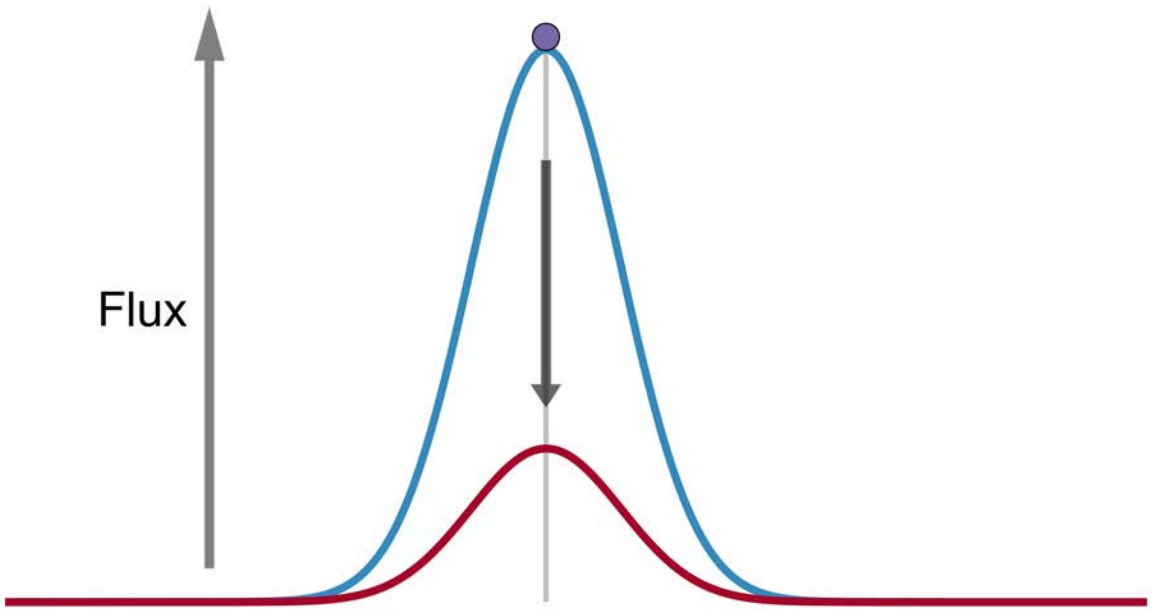
***Measure and record transmit power ($P_{max}$)***

***Reduce transmit power until $N_{pe}$ ~ 0 again***

**Alternatively, insert ND filters in front of detector until $N_{pe}$ ~ 0.**

***Measure and record transmit power ($P_{min}$)***

**Alternatively, record the value of ND filter required to get $N_{pe}$ ~ 0**



# Divergence Estimate Example
# Stafford: Single Scan Lageos1

Stafford data divergence estimation from one satellite example: Lageos 1
Range and cross-section are not constant (Done at 10/21/13, 1924 UTC)

Measured AZ half angle in radians

$\theta saz := 40 \cdot 10^{-6}$

$R1 := 6580$ $R2 := 6337$ One way slant range in km

$\sigma 1 := 15 \cdot 10^{6}$ $\sigma 2 := 15 \cdot 10^{6}$ LRCS in square meters

$Pmax := 2.61$ Max and Min power levels in watts

$Pmin := 0.23$

$$\theta t\_sqr\_az := \left| \frac{-2 \cdot \left(\theta saz^2\right)}{\ln\left[\left(\frac{\sigma 2 \cdot R1^4}{\sigma 1 \cdot R2^4}\right) \cdot \left(\frac{Pmin}{Pmax}\right)\right]} \right|$$

$Div\_full\_az := 2 \cdot \sqrt{\theta t\_sqr\_az}$ Full angle divergence in radians, 1/e2

$$Div\_full\_az = 7.495 \times 10^{-5}$$

Assume the range and cross-section were constant, use AZ here

$$\theta t\_sqr\_az := \left| \frac{-2 \cdot \left(\theta saz^2\right)}{\ln\left(\left(\frac{Pmin}{Pmax}\right)\right)} \right|$$

$Div\_full\_az := 2 \cdot \sqrt{\theta t\_sqr\_az}$ Full angle divergence in radians, 1/e2

$$Div\_full\_az = 7.259 \times 10^{-5}$$

# Stafford single scan measurements taken on two passes of Lageos1

| Date | Satellite | Step size | AZ steps | EL steps | Elevation | Power ratio |
|---|---|---|---|---|---|---|
| 10/21 | Lageos1 | 5 μrad | 16 | 14 | 55° | 0.088 |
| 10/21 | Lageos1 | 5 μrad | 18 | 16 | 70° | 0.026 |
| 10/21 | Lageos1 | 5 μrad | 18 | 18 | 17° | 0.286 |
| 10/21 | Lageos1 | 5 μrad | 17 | 17 | 29° | 0.140 |
| 10/21 | Lageos1 | 5 μrad | 17 | 14 | 32° | 0.187 |
| 10/21 | Lageos1 | 5 μrad | 16 | 18 | 31° | 0.118 |

# Stafford Lageos1 single scan results

| Θ full, R variable | Θ full, R constant |
|---|---|
| 74.95 | 72.59 |
| 66.47 | 66.64 |
| 131.4 | 113.8 |
| 91.3 | 85.74 |
| 90.92 | 92.80 |
| 76.06 | 77.41 |

Comparison of full angle divergence in μradians calculated from AZ scan using range data (left column) and assuming range ~ constant (right column).

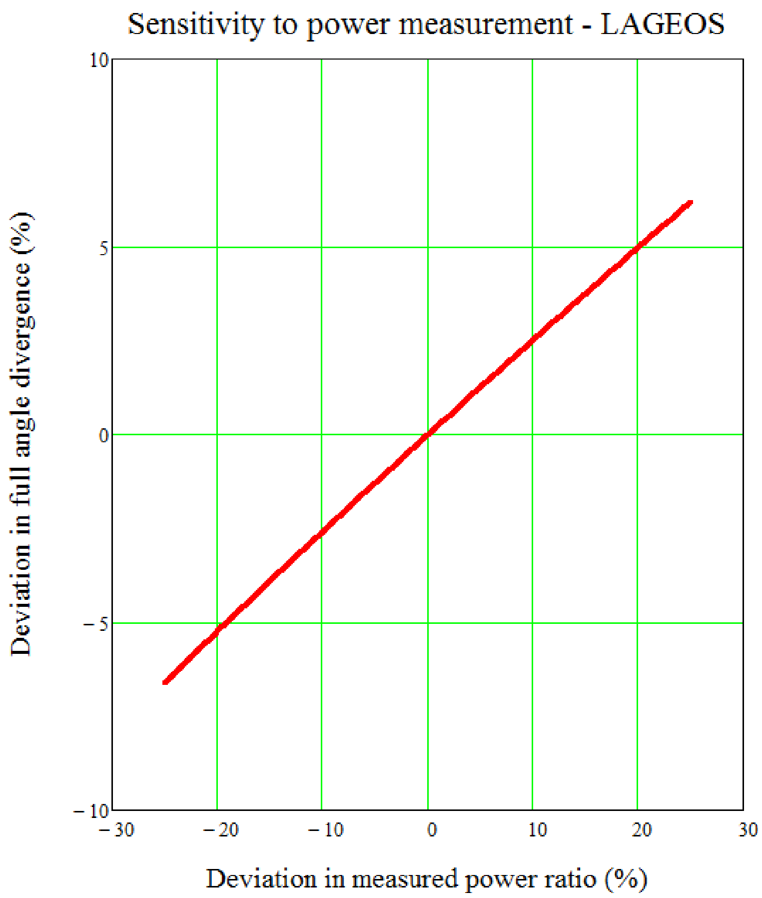
| Θ, AZ scan | Θ, EL scan |
|---|---|
| 72.59 | 63.52 |
| 66.64 | 59.24 |
| 113.8 | 113.8 |
| 85.74 | 85.74 |
| 92.80 | 76.42 |
| 77.41 | 87.08 |

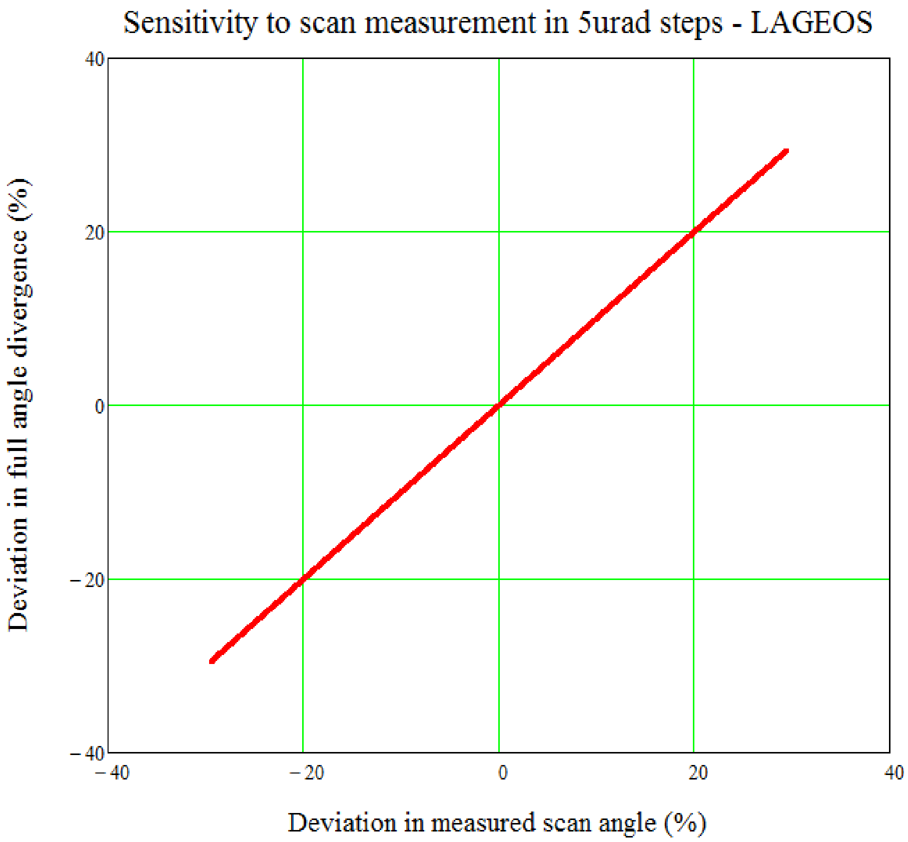
| Mean | StDev |
|---|---|
| 82.89 μrad | 17.63 μrad |

Results of scans on Lageos1 assuming that LRA cross section and range were approximately constant. Divergence mean and stdev calculated from both AZ and EL scans. Sky conditions during ranging were partially overcast obscuring portions of each pass. This could account for the large standard deviation.

Note that first two measurements (lowest divergence) were from a pass with EL > 50 degrees, while last four were from pass less than 32 degrees. The largest divergence measured was at the lowest elevation of 17 degrees.

# Stafford Lageos1 single scan sensitivity analysis




Sensitivity analysis done on Lageos1 measurement number four at 20 degrees elevation. Note that an error in the measured power ratio of 25% results in an error in the estimated divergence of only ~ 7%, while a scan error of 25% results in 25% divergence estimate error.

# NERC single scan measurements

Single scan **measurement method** implemented at NERC and **initial data** taken.

**Table 1** – Satellite scans and ND measurements taken at SGF so far.

| Date | Satellite | Elevation | Laser | Beam | Offsets | | ND |
|---|---|---|---|---|---|---|---|
| | | | | | az. | el. | |
| 06 Oct | Lageos-2 | 40° | 12 Hz | b30 | 17” | 16” | 6 |
| 06 Oct | Beacon-C | 35° | 2 kHz | b37 | 38” | 38” | 9 |
| 06 Oct | Ajisai | 33° | 2 kHz | b37 | 36” | 39” | 10 |
| 07 Oct | Lageos-1 | 56° | 2 kHz | b12 | 27” | 18” | 11 |
| 07 Oct | Lares | 35° | 2 kHz | b25 | 38” | 34” | 9 |
| 07 Oct | Glonass 101 | 41° | 12 Hz | b25 | 18” | 11” | 7 |
| 07 Oct | Glonass 118 | 72° | 12 Hz | b25 | 23” | 19” | 11 |
| 07 Oct | Glonass 118 | 72° | 12 Hz | b35 | 19” | 19” | 9 |

- Two lasers tested: Nd:YAG (12 Hz) and Nd:VAN (2 kHz)
- Various beam expander settings (not directly comparable between lasers)
- Measurements easier to perform on high targets, but maximum beam expansion is limited

# Results from initial NERC single scan measurements

**Discrepancy:** The computed half-angle divergence for 12 Hz laser does not increase with the beam expander settings.

**Possible explanations:**

1) Weather conditions poor with intermittent mist present throughout the night causing measurement errors;
2) Assumption of a perfect Gaussian profile far too optimistic in the presence of turbulence to allow great accuracy from a single measurement…averaging over a larger number of measurements at each divergence setting required;
3) Beam expander is not working exactly as expected due to wear and tear.

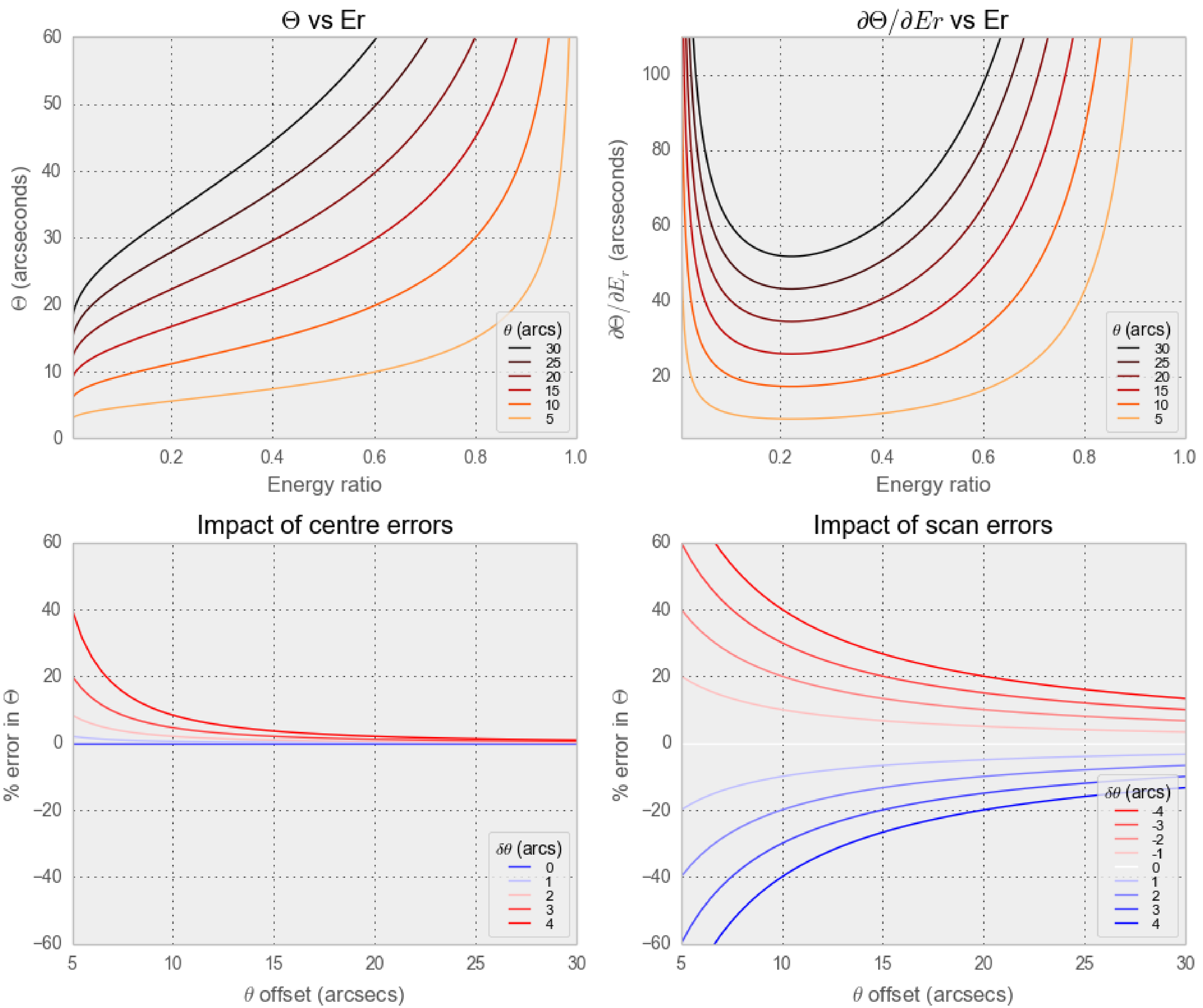
Half-angle divergence in arcseconds for NERC 12 Hz and 2 kHz Lasers

| 12 Hz Laser | 12 Hz Laser | 2 kHz Laser | 2 kHz Laser |
|---|---|---|---|
| Beam expander | $\theta_t$ | Beam expander | $\theta_t$ |
| b25 | 8.5” | b12 | 10” |
| b25 | 9.3” | b25 | 18” |
| b30 | 10.7” | b37 | 18” |
| b35 | 9.5” | b37 | 19” |

1) Used measurements detailed previously;
2) Used average of the azimuth and elevation offsets for each case.

# Error propagation

Error prop. in multiple divergence settings using single scan divergence estimate



# Conclusions and future efforts

A method has been developed which allows the estimation of a station's beam divergence with a scan of a single satellite and a power ratio measurement. Initial measurements have been taken at the U.S. Naval Research Laboratory's Stafford SLR site and at NERC Hertmonceux's SLR site. The results from the initial measurements seem promising and appear to offer a reasonably simple method to measure SLR station divergence.

Future work:

1) Get more stations involved in taking data sets to test the effectiveness of the method.
2) Take larger data sets at each station with identical settings to look at average divergence estimation and statistics.